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MORE ABOUT πgη-CLOSED SETS

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Abstract: In this paper, we study πgη-closed sets in topological spaces and investigate the relationship with other existing generalized closed sets. Moreover, we also study the concepts of πgη-continuous and almost πgη-continuous functions in topological spaces. We obtain some properties of πgη-closed sets and almost πgη-continuous functions. A subset A of a space (X, ℑ) is said to be πgη-closed if η-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X. A function f : X → Y is called πgη-continuous if $f^{-1}(F)$ is πgη-closed in X for every closed set F of Y. A function f : X → Y is called almost πgη-continuous if $f^{-1}(F)$ is πgη-closed in X for every regular closed set F of Y.

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1. Introduction

In 1937, Stone [**15**] introduced the notion of regular open sets. In 1963, Levine [**8**] introduced the concept of semi-open sets. In 1965, Njastad [**13**] introduced the concept of α-open sets. In 1968, the notion of π-open sets were introduced by Zaitsev [**18**] which are weak form of regular open sets and Singal and Singal [**14**] introduced the concept of almost continuous mappings. In 1970, Levine [**9**] initiated the study of generalized closed (briefly g-closed) sets. In 1973, Carnahan [**5**] introduced the concept of R-map. In 1974, Arya and Gupta [**2**] introduced the notion of completely continuity. In 1980, Noiri [**12**] introduced the concept of δ-continuity. In 1990, Munshi [**11**] introduced the concept of super continuity. Arya and Nour [**3**] introduced the notion of gs-closed sets. In 1994, Maki et al. [**10**] introduced the notion of αg-closed sets. In 2000, Dontchev and Noiri [**6**] introduced the notion of πg-closed sets, π-continuous and almost π-continuous functions. In 2006, Aslim and Noiri [**4**] introduced πgs-closed sets in topological spaces, In 2007, Arockiarani and Janaki [**1**] introduced the notion of πgα-closed sets in topological spaces. In 2019, Subbulakshmi, Sumathi, Indirani [**16, 17**] introduced and investigated the notion of η-open and gη-closed sets. In 2021, Kumar [**7**] introduced the concepts of πgη-closed sets and some related topics as πgη-continuous and almost πgη-continuous and obtained some properties of πgη-closed sets.

2. Preliminaries

Throughout this paper, spaces (X, ℑ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** [15] (resp. **regular closed [15]**) if A = int(cl(A)) (resp. A = cl(int(A)). The finite union of regular open sets is said to be **π-open [18]**. The complement of a π-open set is said to be **π-closed**.

Definition 2.1. A subset A of a topological space (X, ℑ) is said to be
(i) **semi-open** (briefly **s-open)** [**8**] if A ⊂ cl(int(A)).
(ii) **α-open** [**13**] if A ⊂ int(cl(int(A))).
(ii) **η-open** [**16**] if A ⊂ in(cl(int(A))) ∪ cl(int(A)).
(iii) **η-closed** [**16**] if A ⊃ cl(int(cl(A))) ∪ int(cl(A)).
The complement of a **s-open** (resp. **α-open)** set is called **s-closed** (resp. **α-closed)**. The intersection of all s-closed (resp. α-closed, η-closed) sets containing A, is called s-closure (resp. **α-closure, η-closure**) of A, and is denoted by **s-cl(A)** (resp. **α-cl(A), η-cl(A))**. The **η-interior** of A, denoted by **η-int(A)** is defined as union of all η-open sets contained in A. We denote the family of all η-open (resp. η-closed) sets of a topological space by **η-O(X)** (resp. **η-C(X)**).

Definition 2.2. A subset A of a space (X, ℑ) is said to be
(1) **generalized closed** (briefly **g-closed**) [**9**] if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(2) **πg-closed** [**6**] if cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(3) **generalized s-closed** (briefly **gs-closed**) [**3**] if s-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.

(4) **πgs-closed** [4] if s-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(5) **α-generalized closed** (briefly **αg-closed**) [**10**] if α-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(6) **πgα-closed** [**1**] if α-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(7) **generalized η-closed** (briefly **gη-closed**) [**17**] if η-cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(8) **πgη-closed** [**7**] if η-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
(9) **g-open** (resp. **πg-open**, **gs-open, πgs-open, αg-open, πgα-open, gη-open, πgη-open**) set if the complement of A is g-closed (resp. πg-closed, gs-closed, πgs-closed, αg-closed, πgα-closed, gη-closed, πgη-closed).

Remark 2.3. From the above definitions and known results the relationship between πgη-closed sets and some other existing generalized closed sets are implemented in the following Figure:

closed	⇒	g-closed	⇒	πg-closed
⇓		⇓		⇓
α-closed	⇒	αg-closed	⇒	πgα-closed
⇓		⇓		⇓
s-closed	⇒	gs-closed	⇒	πgs-closed
⇓		⇓		⇓
η-closed	⇒	gη-closed	⇒	πgη-closed

Where none of the implications is reversible as can be seen from the following examples:

Example 2.4. Let X = {a, b, c, d, e} and ℑ = {ϕ, {b}, {b, c}, {a, d}, {a, b, d}, {a, b, c, d}, X}. Then A = {b} is πgs-closed as well as πgη-closed. But it is neither gs-closed nor πg-closed.

Example 2.5. Let X = {a, b, c, d, e} and ℑ = {ϕ, {b}, {b, c}, {a, d}, {a, b, d}, {a, b, c, d}, X}. Then A = {a} and B = {b} are πgs-closed as well as πgη-closed but not closed.

Example 2.6. Let X = {a, b, c, d} and ℑ = {ϕ, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}, X}. Then A = {a, c, d} and B = {b, c, d} are πgs-closed as well as πgη-closed but not closed.

Example 2.7. Let X = {a, b, c, d} and ℑ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. Then A = {a, b} is πgα-closed, πgs-closed as well as πgη-closed. But it is neither closed nor αg-closed.

Example 2.8. Let X = {a, b, c, d} and ℑ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Then A = {a, b, c} and B = {a, b, d} are πg-closed, πgα-closed, πgs-closed as well as πgη-closed but not closed.

Example 2.9. Let X = {a, b, c, d} and ℑ = {ϕ, {a}, {d}, {a, d}, {c, d}, {a, c, d}, X}. Then A = {c} is πgα-closed as well as πgη-closed. But it is neither closed nor g-closed.

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Example 2.10. Let X = {a, b, c, d} and ℑ = {ϕ, {c}, {d}, {c, d}, {b, c, d}, X}. Then A = {b} is g-closed, αg-closed, gη-closed, πgα-closed, πgs-closed, πgη-closed. But it is closed.

Example 2.11. Let X = {a, b, c} and ℑ = {ϕ, {a}, {c}, {a, c}, X}. Then A = {c} is η-closed as well as πgη-closed but not α-closed.

Example 2.12. Let X = {a, b, c} and ℑ = {ϕ, {a}, {b, c}, X}. Then A = {a, b} is gη-closed as well as πgη-closed but not closed.

3. πgη-closure and πgη-interior

Definition 3.1. For any set A ⊂ X, the **πgη-closure** of A is defined as the intersection of all πgη-closed sets containing A. We write

$$\pi g\eta\text{-cl}(A) = \{F : A \subset F \text{ and } F \text{ is } \pi g\eta\text{-closed set in } X\}$$

Definition 3.2. The **πgη-interior** of A, denoted by **πgη-int(A)** is defined as union of all πgη-open sets contained in A. We write

$$\pi g\eta\text{-int}(A) = \{G : A \supset G \text{ and } G \text{ is } \pi g\eta\text{-open set in } X\}$$

Lemma:3.3. For an x ∈ X, x ∈ πgη-cl(A) if and only if V ∩ A ≠ ϕ for every πgη-open set V containing x.
Proof. First, let us suppose that there exists a πgη-open set V containing x such that V ∩ A = ϕ. Since A ⊂ X – V, πgη-cl(A) ⊂ X – V ⇒ x ∉ πgη-cl(A), which is a contradiction to the fact that x ∈ πgη-cl(A). Hence V ∩ A ≠ ϕ for every πgη-open set V containing x.

On the other hand, let x ∉ πgη-cl(A). Then there exists a πgη-closed subset F containing A such that x ∉ F. Then x ∈ X – F and X – F is πgη-open. Also, $(X - F) \cap A \neq \phi$, a contradiction. Hence the lemma.

Lemma 3.4. Let A and B be subsets of $(X, \mathfrak{I})$. Then
(i) πgη-cl(ϕ) = ϕ and πgη-cl(X) = X.
(ii) If $A \subset B$, then πgη-cl(A) ⊂ πgη-cl(B).
(iii) $A \subset$ πgη-cl(A).
(iv) πgη-cl(A) = πgη-cl(πgη-cl(A)).
(v) πgη-cl(A∪B) = πgη-cl(A) ∪ πgη-cl(B).

Proof. Obvious.

Remark 3.5. If $A \subset X$ is πgη-closed, then πgη-cl(A) = A. But the converse is need not be true as seen in following example.

Example 3.6. Let X = {a, b, c, d} and $\mathfrak{I}$ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}}. Let A = {a}, πgη-cl(A) = {a} = A , but A = {a} is not πgη-closed in X.

Lemma 3.7. Let A and B be subsets of X. Then πgη-cl(A ∩ B) ⊂ πgη-cl(A) ∩ πgη-cl(B).
Proof. Since A ∩ B ⊂ A, B.
⇒ πgη-cl(A ∩ B) ⊂ πgη-cl(A), πgη-cl(A ∩ B) ⊂ πgη-cl(B)
⇒ πgη-cl(A ∩ B) ⊂ πgη-cl(A) ∩ πgη-cl(B).

Remark 3.8. The converse of the above need not be true as seen in following example.

Example 3.9. Let X = {a, b, c, d, e} and $\mathfrak{I}$ = {ϕ, X, {a}, {e}, {a, e}, {c, d}, {a, c, d}, {a, c, d, e}, {c, d, e}}. Let A = {a, c, e} ⊂ X, B = {d} ⊂ X. Then πgη-cl(A) = {a, b, c, e}, πgη-cl(B) = {b, d}. But πgη-cl(A) ∩ πgη-cl(B) = {b} ⊄ πgη-cl(A ∩ B).

Remark 3.10. We denote πgη-closed sets in topological space by πgη-C(X) and πgη-open sets by πgη-O(X).

Definition 3.11. $\mathfrak{I}^*_{\pi g\eta}$ = {V ⊂ X : πgη-cl(X – V) = X – V}

Theorem: 3.12. If πgη-O(X) is topology, then $\mathfrak{I}^*_{\pi g\eta}$ is a topology.

Proof.

Clearly, ϕ, X ∈ $\mathfrak{I}^*_{\pi g\eta}$. Let $\{A_i : i \in A\} \in \mathfrak{I}^*_{\pi g\eta}$.

$$\begin{aligned} \pi g\eta\text{-cl}(X - (\cup A_i)) &= \pi g\eta\text{-cl}(\cap (X - A_i)) \\ &\subset \cap\, \pi g\eta\text{-cl}(X - A_i) \\ &= \cap (X - A_i) \\ &= X - \cup A_i \end{aligned}$$

Hence $\cup A_i \in \mathfrak{I}^*_{\pi g\eta}$.

Let A, B ∈ $\mathfrak{I}^*_{\pi g\eta}$.

$$\begin{aligned} \text{Now, } \pi g\eta\text{-cl}(X - (A \cap B) &= \pi g\eta\text{-cl}((X - A) \cup (X - B)) \\ &= \pi g\eta\text{-cl}(X - A) \cup \pi g\eta\text{-cl}((X - B) \\ &= (X - A) \cup (X - B) \end{aligned}$$

Thus A ∩ B ∈ $\mathfrak{I}^*_{\pi g\eta}$ and hence $\mathfrak{I}^*_{\pi g\eta}$ is a topology.

Definition 3.13. Let X be a topological space and let x ∈ X. A subset N of X is said to be **πgη-neighbourhood** of x if there exists a πgη-open set G such that x ∈ G ⊂ N.

Definition. 3.14. Let A be a subset of X. A point x∈ A is said to be **πgη-interior** point of A if A is a πgη-nbhd of x. The set of all πgη-interior of A and is denoted by πgη-int(A).

Theorem: 3.15. If A be a subset of X. Then πgη-int(A) = ∪ {G : G is πgη-open, G ⊂ A}

Proof. Straight forward.

Theorem 3.16. Let A and B be subsets of X. Then
(i) πgη-int(X) = X and πgη-int(ϕ) = ϕ.
(ii) πgη-int(A) ⊂ A.
(iii) If B is any πgη-open set contained in A, then B ⊂ πgη-int(A).
(iv) If A ⊂ B, then πgη-int(A) ⊂ πgη-int(B).
(v) πgη-int(πgη-int(A)) = πgη-int(A).

Proof. Straight forward.

Theorem 3.17. If a subset A of a space X is πgη-open, then πgη-int(A) = A.

Proof. Obvious.

Remark 3.18. The converse of the above need not be true as seen in the following example.

Example 3.19. Let X = {a, b, c, d} and ℑ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}}. Let A = {c, d}. Then πgη-int(A) = {c, d} = A. But A = {c, d} is not πgη-open.

Theorem 3.20. If A and B are subsets of X, then πgη-int(A) ∪ πgη-int(B) ⊂ πgη-int(A ∪ B).

Proof. We know that A ⊂ A ∪ B and B ⊂ A ∪ B. Then πgη-int(A) ⊂ πgη-int(A∪B), πgη-int(B) ⊂ πgη-int(A ∪ B). Hence πgη-int(A) ∪ πgη-int(B) ⊂ πgη-int(A ∪ B).

Theorem 3.21. If A and B are subsets of a space X, then πgη-int(A ∩ B) = πgη-int(A) ∩ πgη-int(B)

Proof. We know that A ∩ B ⊂ A , A ∩ B ⊂ B. Then πgη-int(A ∩ B) ⊂ πgη-int(A) and πgη-int(A ∩ B) ⊂ πgη-int(B).
⇒ πgη-int(A ∩ B) = πgη-int(A) ∩ πgη-int(B)…………….(1)
Again, let x ∈ πgη-int(A) ∩ πgη-int(B). Then x ∈ πgη-int(A) and x ∈ πgη-int(B). Hence x is a πgη-interior point of each of sets A and B. It follows that A and B are πgη-nbhds of x, so that their intersection A ∩ B is also a πgη-nbhd of x. Hence x ∈ πgη-int(A ∩ B).

Thus, x∈ πgη-int(A) ∩ πgη-int(B) ⇒ x ∈ πgη-int(A ∩ B).
Therefore, πgη-int(A) ∩ πgη-int(B) ⊂ πgη-int(A ∩ B)……………(2)
From (1) and (2), πgη-int(A ∩ B) = πgη-int(A) ∩ πgη-int(B).

Theorem 3.22. If A is subset of X, then
(i) η-int(A) ⊂ πgη-int(A) and
(ii) (X – πgη-int(A)) = πgη-cl(X – A) and (X – πgη-cl(A)) = πgη-int(X–A).

Proof. Straight forward.

4. πgη-continuous and almost πgη-continuous functions

Definition 4.1. A function f : X → Y is called:
(a) **R-map** [5] if f^{-1}(F) is regular closed in X for every regular closed set F of Y.
(b) **completely continuous** [2] (resp. **super continuous** [11]) if f^{-1}(F) is regular closed (resp. δ-closed) in X for every closed set F of Y.
(c) **π-continuous** [6] (resp. **η-continuous [17], gη-continuous [17], πgη-continuous** [7]) if f^{-1}(F) is π-closed (resp. η-closed, gη-closed, πgη-closed) in X for every closed set F of Y.
(d) **almost π-continuous** [6] (resp. **δ-continuous** [12], **almost continuous** [14], **almost η-continuous**, **almost gη-continuous**, **almost πgη-continuous [7]**) if f^{-1}(F) is π-closed (resp. δ-closed, closed, η-closed, gη-closed, πgη-closed) in X for every regular closed set F of Y.
(e) **πgη-irresolute** [7] if f^{-1}(F) is πgη-closed in X for every πgη-closed set F of Y.

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Example 4.2. Let X = {x, y, z}, ℑ = {ϕ, X, {x}, {x, y}} and ρ = {ϕ, X, {x}, {x, z}}. The identity function f : (X, ℑ) → (Y, ρ) is an R-map as well as almost π-continuous. It is also πgη-continuous and almost πgη-continuous but not continuous.

Example 4.3. Let X = {x, y, z}, ℑ = {ϕ, X, {x}, {y}, {x, y}, {x, z}} and ρ = {ϕ, X, {x}, {y}, {x, y}}. The identity function f : (X, ℑ) → (Y, ρ) is continuous as well as η-continuous but not δ-continuous.

Example 4.4. Let X = {x, y, z}, ℑ = {ϕ, X, {x}, {y}, {x, y}} and define f : (X, ℑ) → (X, ℑ) as follows: f(x) = f(y) = x and f(z) = z. Then f is π-continuous, πgη-continuous and almost πgη-continuous but not R-map.

Example 4.5. Let X = {x, y, z, w} and ℑ = {ϕ, X, {x}, {y}, {x, y}, {x, y, z}, {x, y, w}}. Define f : (X, ℑ) → (X, ℑ) as follows: f(x) = z, f(y) = x, f(z) = y and f(w) = z. Then f is almost πgη-continuous.

Example 4.6. Let X = {x, y, z}, ℑ = {ϕ, X, {x}}, Y = {a, b} and ρ = {ϕ, Y, {a}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = f(z) = b and f(y) = x. Then f is gη-continuous as well as πgη-continuous. It is almost πgη-continuous but not continuous.

Example 4.7. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {x}, {z}, {x, z}} and ρ = {ϕ, Y, {x}, {y}, {x, y}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = x, f(y) = z and f(z) = y. Then f^{-1}({z}) = {y}, f^{-1}({x, z}) = {x, y}, f^{-1}({y, z}) = {y, z}. Therefore, f is gη-continuous as well as πgη-continuous. It is almost πgη-continuous.

Example 4.8. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {x}, {z}, {x, z}} and ρ = {ϕ, Y, {y, z}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = x, f(y) = z and f(z) = y. Then f is gη-continuous as well as πgη-continuous. It is almost πgη-continuous

Example 4.9. Let X = Y = {x, y, z}, ℑ = {ϕ, X, {y, z}} and ρ = {ϕ, Y, {x}}. Define f : (X, ℑ) → (Y, ρ) as follows: f(x) = y, f(y) = z and f(z) = x. Then f is gη-continuous as well as πgη-continuous. It is almost πgη-continuous

Example 4.10. Let $X = Y = \{x, y, z\}$, $\Im = \{\phi, X, \{x\}, \{z\}, \{x, z\}\}$ and $\rho = \{\phi, Y, \{x\}, \{y\}, \{x, y\}\}$. Define $f : (X, \Im) \to (Y, \rho)$ as follows: $f(x) = x$, $f(y) = z$ and $f(z) = y$. Then $f^{-1}(\{x\}) = \{x\}$, $f^{-1}(\{y\}) = \{z\}$, $f^{-1}(\{z\}) = \{y\}$, $f^{-1}(\{x, z\}) = \{x, y\}$, $f^{-1}(\{y, z\}) = \{y, z\}$. Since inverse image of every gη-open set in Y is gη-open in X. Therefore, f is gη-continuous as well as πgη-continuous. It is almost πgη-continuous

Remark 4.2. From the above definitions, proposition and known results, we have following diagram:

completely continuous	⇒	R-map
⇓		⇓
π-continuous	⇒	almost π-continuous
⇓		⇓
super continuous	⇒	δ-continuous
⇓		⇓
continuous	⇒	almost continuous
⇓		⇓
η-continuous	⇒	almost η-continuous
⇓		⇓
gη-continuous	⇒	almostg gη-continuous
⇓		⇓
πgη-continuous	⇒	almostg πgη-continuous

Where none of the implications is reversible as can be seen from the above examples:

5. Properties of πgη-continuous functions

Definition 5.1. A topological space X is called a **πgη-$T_{1/2}$ space [7]** if every πgη-closed set is η-closed.

Theorem 5.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then the following are equivalent:
(a) f is πgη-continuous.
(b) The inverse image of every open set in Y is πgη-open in X.

Proof. Follows from the definitions.

Theorem 5.3. If $f : (X, \tau)\to(Y, \sigma)$ is πgη-continuous, then $f(\pi g\eta\text{-}cl(A)) \subset cl(f(A))$ for every subset A of X.

Proof. Let $A \subset X$. Since f is πgη-continuous and $A \subset f^{-1}(cl(f(A)))$, we obtain $\pi g\eta\text{-}cl(A) \subset f^{-1}(cl(f(A))$ and then $f((\pi g\eta\text{-}cl(A)) \subset cl(f(A))$.

Remark 5.4. The converse of the above need not be true as seen in following example.

Example 5.5. Let $X = \{a, b, c, d\}$, $\Im = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{c, d\}\}$. Let $f : (X, \Im) \to (X, \sigma)$ be an identity map. Let $A = \{a, b\}$. Then $\pi g\eta\text{-}cl(\{a, b\}) = \{a, b\} \subset f^{-1}(cl(f(\{a,b\}))) = X$. But $f^{-1}(\{a, b\}) = \{a, b\}$ is not πgη-closed in X. Hence f is not πgη-continuous.
Proposition 5.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a πgη-continuous function and H be π-open, πgη-closed subset of X. Assume that πgη-C(X, τ) closed under finite intersections. Then the restriction be $f/H : (H, \tau/H) \to (Y, \sigma)$ is πgη-continuous.
Proof. Let F be any η-closed subset in Y. By hypothesis and our assumption $f^{-1}(F) \cap H_1$, it is πgη-closed in X. Since $(f/H)^{-1}(F) = H_1$, it is sufficient to show that H_1 is πgη-closed in H. Let $H_1 \subset G_1$ is any π-open set in H. We know that a subset A of X is open, then $\pi\text{-}O(A, \tau/A) = \{V \cap A : V \in \pi O\text{-}(X,\tau\}$…………(1). By (1), $G_1 = G \cap H$ for some π-open set G in X.
Then $H_1 \subset G_1 \subset G$ and H_1 is πgη-closed in X implies $\eta\text{-}cl_X(H_1) = \eta\text{-}cl_X(H_1) \cap H \subset G \cap H = G_1$ and so H_1 is πgη-closed in H. Therefore, f/H is πgη-continuous.

Generalization of Pasting Lemma for πgη-continuous functions.

Theorem 5.7. Let $X = G \cup H$ be a topological space with topology τ and Y be a topological space with topology σ. Let $f : (G, \tau/G) \to (Y, \sigma)$ and $g : (H, \tau/H) \to (Y, \sigma)$ be πgη-continuous functions such $f(x) = g(x)$ for every $x \in G \cap H$. Suppose that both G and H are π-open and πgη-closed in X. Then their combination $(f \nabla g) : (X, \tau) \to (Y, \sigma)$ defined by $(f \nabla g)(x) = f(x)$ if $x \in G$ and $(f \nabla g)(x) = g(x)$ if $x \in H$ is πgη-continuous.

Proof. Let F be any closed set in Y. Clearly $(f \nabla g)^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. Since $f^{-1}(F)$ is πgη-closed in G and G is π-open in X and πgη-closed in X, $f^{-1}(F)$ is πgη-closed in X. Similarly, $g^{-1}(F)$ is π-open in X. Therefore, $(f \nabla g)$ is πgη-continuous.

Proposition 5.8. If a function $f : (X, \tau) \to (Y, \sigma)$ is πgη-irresolute, then
(i) $f(\pi g\eta\text{-}cl(A)) \subset \pi g\eta\text{-}cl(f(A))$ for every subset A of X.
(ii) $\pi g\eta\text{-}cl(f^{-1}(B)) \subset f^{-1}(\pi g\eta\text{-}cl(B))$ for every subset B of Y.

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Proof.
(i) For every A ⊂ X, πgη-cl(f(A)) is πgη-closed in Y. By hypothesis, f^{-1}(πgη-cl(A)) is πgη-closed in X. Also, A ⊂ f^{-1}(f(A)) ⊂ f^{-1}(πgη-cl(A)). By the definition of πgη-closure, we have πgη-cl(A) ⊂ f^{-1}(πgη-cl(A)). Hence, we get f(πgη-cl(A)) ⊂ πgη-cl(f(A)).

(ii) πgη-cl(B) is πgη-closed in Y and so by hypothesis, f^{-1}(πgη-cl(B)) is πgη-closed in X. Since f^{-1}(B) ⊂ f^{-1}(πgη-cl(B)), it follows that πgη-cl(f^{-1}(B)) ⊂ f^{-1}(πgη-cl(B)).

Theorem: 5.9. For a function f : X → Y, the following are equivalent to one another:
(i) f is almost πgη-continuous.
(ii) f^{-1}(V) is πgη-open in X for every regular open set V of Y.
(iii) f^{-1}(int(cl(V))) is πgη-open in X for every open set V of Y.
(iv) f^{-1}(cl(int(V))) is πgη-closed in X for every closed set V of Y.

Proof.
(i) ⇒ (ii). Let V be a regular open subset of Y. Since Y – V is regular closed and f is almost πgη-continuous, then f^{-1}(Y – V) = X – f^{-1}(V) is πgη-closed in X. Thus f^{-1}(V) is πgη-open on X

(ii) ⇒ (i). Let V be a regular closed subset of Y. Then Y – V is regular open. By hypothesis, f^{-1}(Y – V) = X – f^{-1}(V) is πgη-open in X. Then f^{-1}(V) is πgη-closed and hence f is almost πgη-continuous.

(ii) ⇒ (iii). Let V be an open subset of Y. Then int(cl(V)) is regular open. By hypothesis f^{-1}(int(cl(V)) is πgη-open in X.

(iii) ⇒ (ii). Let V be a regular open subset of Y. Since Y – int(cl(V))) and every regular open set is open, then f^{-1}(V) is πgη-open in X.

(iii) ⇒ (iv). Let V be a closed subset of Y. Then Y – V is open. By hypothesis, f^{-1} (int(cl(Y – V))) = f^{-1}(Y – cl(int(V))) = X– f^{-1}(Y–cl(int(V))) is πgη-open in X. Hence f^{-1}(cl(int(V))) is πgη-closed in X.

(iv) ⇒ (iii). Let V be a open subset of Y. Then Y – V is closed. By hypothesis, f^{-1}(cl(int(Y – V))) = f^{-1}(Y – cl(int(V))) = X – f^{-1}(int(cl(V))) is πgη-closed in X. Hence f^{-1}(int(cl(V))) is πgη-open in X.

Proposition 5.10. If f is πgη-irresolute, then it is almost-πgη-continuous.

Proof. Straight forward.

Definition 5.11. A function f : (X, τ)→(Y, σ) is called a **π-open map** [**6**] if f(U) is π-open in (Y, σ) for every π-open set U in (X, τ).
Proposition 5.12. If f is bijective, π-open, almost-πgη-continuous, then f is πgη-irresolute.

Proof. Let F be a πgη-closed set of Y. Let f^{-1}(F) ⊂ U, where U is π-open in X. Then F ⊂ f(U). Since f is π-open, f(U) is π-open in Y, F is πgη-closed set in Y and F ⊂ f(U) ⇒ η-cl(F) ⊂ f(U). (i. e) f^{-1}(η-cl(F)) ⊂ U. Since f is almost-πgη-continuous, η-cl(f^{-1}(η-cl(F))) ⊂ U.
So, η-cl(f^{-1}(F)) ⊂ η-cl(f^{-1}(η-cl(F))) ⊂ U.
⇒ f^{-1}(F) is πgη-closed in X. Hence f is πgη-irresolute.

Proposition 5.13. If f is bijective, π-open, R-map, then f is πgη-irresolute.

Proof. Since f is an R-map, it is almost πgη-continuous. By **Proposition 5.13**, f is πgη-irresolute.

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